Numerical solution of the Smoluchowski equation for a vibrofluidized granular bed

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A stochastic approach, similar to that used to describe Brownian motion, was used to model the displacement probability of grains in a three-dimensional vibrofluidized granular bed. As neither an analytical description nor measurements of the diffusion coefficients were available, the governing partial differential equation, namely, the Smoluchowski equation, was solved numerically using an iterative procedure, modifying the granular temperature profile at each step. The results of this stochastic model were compared to experimental measurements of the displacement probability density made using positron emission particle tracking. The results indicate that methods based on hard elastic systems such as the Smoluchowski equation are appropriate to granular systems, particularly over timescales greater than the mean collision time.

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I. INTRODUCTION

Granular materials have been studied closely for over a hundred years [1]. The ubiquity of such materials in both nature and industry has precipitated many studies with the aim of improving transport and storage methods. Recently, the complexity and richness in the behavior of dissipative and nonequilibrium systems has led to renewed interest in granular flows at the microscopic or single particle level [2]. On the whole, the investigations have centered on theoretical and numerical analysis of the problem. For example, Savage and Jenkins have pioneered the use of kinetic theory methods to model microscopic motion in granular flow [3] and McNamara and Luding have studied dissipative systems using molecular dynamics and event driven simulations [4]. The need for experimental validation of these approaches was emphasized by Campbell in his review paper [5]. In response to this, a number of phenomena have been systematically investigated, including convection [6], gravity driven flows [7], and wave propagation [8].

The success of the analogy between atomic and granular gases is surprising considering the fundamental differences between such systems. Dissipation of the kinetic energy into heat during collisions leads to a ''granular gas'' being, by definition, out of equilibrium. Consequently, energy must be continuously introduced into the system, normally accomplished using boundary motion. The kinetic theory of gases is built upon the foundations of the *equilibrium* theories of thermodynamics and statistical mechanics. Therefore, it is not clear that such an approach is entirely appropriate to a nonequilibrium steady state such as that seen in a vibrated granular bed. Despite these apparent difficulties an understanding of granular flow is being developed as hydrodynamic theories are constructed from microscopic foundations [9].

Until recently, experimental investigations of granular flow at the ''microscopic'' or single particle level were somewhat impractical. However, the development of high speed digital photography techniques has led to a series of investigations on the microscopic behavior of grains in a two-dimensional (2D) vibrated granular bed, which have indicated that ideal granular systems may not be far from equilibrium: Warr *et al.* [10] fitted an Maxwell-Boltzmann distribution function to the measured velocity distribution functions and extracted the granular temperature from such curves. The experimental velocity distributions were found to be in close correspondence to Maxwell-Boltzmann profiles, indicating that such methods are appropriate to vibro fluidized granular beds [11]. A subsequent investigation showed that the dynamic structure of a granular material was in many respects similar to that seen in a thermal liquid [12]. More recently, Chapman-Enskog kinetic theory was tested for a fluidized granular bed by comparing measurements and predictions of the self-diffusion coefficient [13]. The good comparison between practice and theory suggested that kinetic theory approaches are valid, at least for relatively elastic collisions (restitution coefficient, $\varepsilon > 0.9$) up to packing fractions of about 60% [13].

The availability of whole field techniques for the analysis of rapidly evolving three-dimensional systems, such as vibrated granular beds, is limited, e.g., techniques such as high speed photography are unable to probe beyond the surface of a three-dimensional bed. A number of techniques have recently been developed to observe the motion of grains in three-dimensional systems. Magnetic resonance imaging has been used for medical applications for some time and has recently been adapted for granular flows. In particular, it was used to observe convection rolls in tapped and vibrated granular columns [14]. Diffusive wave spectroscopy is an alternative technique that is also noninvasive [15]. It has found applications in probing the mean motion in gas fluidized beds [16] where it was also found that kinetic theories were broadly valid even in this multiphase system. Positron emission tomography (PET) has been used for over 20 years to study quasistatic flow fields [17]. From this, positron emission particle tracking (PEPT) developed allowing the tracking of a single radioactive tracer particle at high resolution in three-dimensional geometries. This powerful technique has recently been used to investigate a number of ex-

perimental situations, e.g., rotating beds [18] and paste flow [19].

This paper will detail the modeling of a vibrofluidized granular gas using the familiar Smoluchowski equation. One may treat the motion of a grain buffeted by the random motion of neighboring particles by applying Brownian motion theory or more generally using a stochastic approach. A complete analysis of a granular system must involve description of the evolution of the system in both the spatial and velocity spaces. However, when one is interested in times of the order of the mean collision time or greater, such stochastic approaches can be simplified to just the spatial development and the Smoluchowski equation may then be employed [20].

The experimental setup used to examine vibrofluidized beds will be introduced in Sec. II. In Sec. III, the Smoluchowski equation and its derivation will be considered, and then in Sec. IV a finite-difference approximation to the Smoluchowski equation will be presented. In Sec. V the numerical and experimental results will be compared and analyzed to determine the extent to which stochastic approaches may be used to describe granular flows.

II. EXPERIMENT

Positron emission particle tracking was used to follow the motion of a single grain throughout a three-dimensional vibrofluidized granular bed. This experimental technique has been described in detail in previous publications [21,18,22]. However, for completeness, we describe the system briefly as follows. A particle, containing relatively large quantities of oxygen, is irradiated with a highly energetic beam of ^{3}He particles. The interactions between the target nuclei and the incident particles results in the formation of a radioisotope of fluorine, ^{18}F. This radioisotope decays through the emission of positrons, which are annihilated when they encounter electrons. This annihilation results in the conversion of matter into two back-to-back 511-keV photons, which are simultaneously detected in a pair of detectors. This allows the position to be determined by triangulation from successive events.

A three-dimensional granular gas was generated using a Ling Dynamic Systems (LDS) vibration system. A sinusoidal signal was fed through a field power supply (LDS FPS 1) and power amplifier (LDS PA 1000) into a wide frequency band electrodynamic transducer (LDS V651). This system has a frequency range of 5–5000 Hz, a maximum acceleration of 100 g and maximum amplitude of 12.5 mm. A cylindrical cell 145-mm diameter and 300-mm height was placed on the upper surface of the vibrating piston, itself placed between the photon detectors (Fig. 1). The cell was constructed of polymethyl methacrylate to limit the attenuation of the γ rays as they traveled through the experimental apparatus. The cell was vibrated at a frequency of 50 Hz and at an amplitude, A_0 of 1.91 mm. Glass ballotini balls of diameter 5.0 ± 0.2 mm (restitution coefficient, $\varepsilon = 0.91$) were used as the granular medium. A single grain was irradiated to provide the source of γ rays during the experiments.

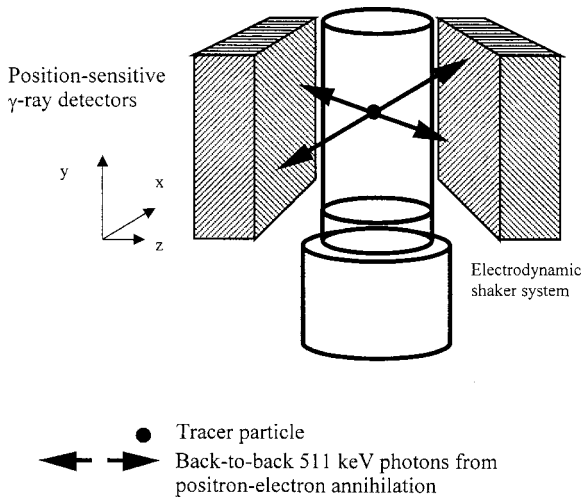

FIG. 1. Positron emission particle tracking and shaker system experimental arrangement.

III. STOCHASTIC DESCRIPTION OF A VIBROFLUIDIZED GRANULAR BED

Analysis of the stochastic behavior of grains in a granular bed, i.e., a probabilistic approach to the transient self-motion, is particularly useful when using experimental techniques such as PEPT where only a single grain is tracked throughout the system. The stochastic phenomenological approach to Brownian motion was developed by Langevin, Einstein, and Smoluchowski [24,23] who were able to formulate a general theory, initially constructed for equilibrium fluids. Considering the remarkable success of equilibrium gas kinetic theories in modeling granular systems, it is important to judge whether the equilibrium stochastic methods developed for Brownian motion can be used to model vibrated granular beds.

A. Continuity and constitutive relations

If we consider a gas of similar particles undergoing rapid motion and successive collisions, and within that gas we ''tag'' a small number of the particles, then if the concentration of the tagged particles is negligible, the motion of the tagged particles is equivalent to that of single particle motion. In this case, the conditional displacement probability density $P(\mathbf{r},t)$, and the current $\mathbf{j}(\mathbf{r},t)$ satisfies the continuity equation [24]

$$\dot{P}(\mathbf{r},t) + \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r},t) = 0 \tag{1}$$

and the Fickian constitutive relation

$$\mathbf{j}(\mathbf{r},t) = -D \boldsymbol{\nabla} P(\mathbf{r},t), \tag{2}$$

where $\mathbf{r}$ is the position vector of a tagged particle, D is the self-diffusion coefficient, and t is the time. Solution of Eqs. (1) and (2) in the low frequency limit, i.e., $t \gg \tau_E$, where τ_E is the mean time between collisions, leads to the result

$$P(\mathbf{r},t) = \frac{1}{(4\pi D t)^{3/2}} \exp\left(-\frac{r^2}{4Dt} \right). \tag{3}$$

The mean squared displacement can be calculated from this probability density function in the usual way (see e.g., Ref. [24]) and leads to the well known Einstein relation

$$D=\lim_{t\to\infty}\frac{1}{6t}\langle|r(t)-r(0)|^2\rangle. \tag{4}$$

However, this analysis is only true for unbounded, homogenous systems, such as an unconstrained thermal gas or fluid. In a granular system, the presence of gravity, as well as causing acceleration between collisions, results in a varying packing fraction profile in the vertical (y) direction, and so D is actually a position dependent quantity. In this situation the constitutive relation (2) is modified to

$$\mathbf{j}(\mathbf{r},t)=-D(\mathbf{r})\boldsymbol{\nabla}P(\mathbf{r},t)-\mathbf{u}(\mathbf{r})P(\mathbf{r},t) \tag{5}$$

and the combination of Eq. (5) with Eq. (1) results in the Smoluchowski equation

$$\dot{P}(\mathbf{r},t)=\boldsymbol{\nabla}\cdot\{D(\mathbf{r})\boldsymbol{\nabla}P(\mathbf{r},t)+\mathbf{u}(\mathbf{r})P(\mathbf{r},t)\}, \tag{6}$$

where $\mathbf{u}(\mathbf{r})$ is the terminal velocity of a particle in a resisting medium, when an external potential field is applied [25].

In the case of a vibrofluidized granular bed, $\mathbf{u}(\mathbf{r})=u(y)$ only, as gravitational forces act only in the vertical direction. Similarly $D(\mathbf{r})=D(y)$, since the granular temperature and packing fraction distributions, on which D depends, are only weak functions of x and z [13,26].

B. Granular temperature

The granular temperature, E_O, is defined in the usual manner,

$$E_O=\tfrac{1}{3}(E_X+E_Y+E_Z)=\tfrac{1}{3}(m\overline{v_X^2}+m\overline{v_Y^2}+m\overline{v_Z^2}), \tag{7}$$

where v_X, v_Y, and v_Z are the particle velocity components resolved in the x, y, or z directions and E_X, E_Y, and E_Z are the granular temperatures for each of the respective directions. It is well known that due to dissipation, the granular temperature in a vibrofluidized bed is anisotropic [10], and that, unlike the case of elastic spheres, equipartition of energy does not hold. In most cases $E_Y>E_X=E_Z$ [10]. This is an important consideration when probing the spatial development of a system as it causes a corresponding anisotropy in the self-diffusion coefficients [13]. For simplicity, however, we treat D as a scalar throughout this investigation.

The relationships between D, E_O, and packing fraction η have been investigated in two dimensions [13]. The Enskog kinetic theory equation was shown to be obeyed to within 10% for packing fractions up to about 0.6 and $\varepsilon=0.92$. In three dimensions, the corresponding Enskog result reads

$$D=\frac{3}{8nd^2g_O(d)}\left(\frac{E_O}{\pi m}\right)^{1/2}, \tag{8}$$

where n is the number density, m is the particle mass, d is the particle diameter, and $g_O(d)$, the radial distribution function at contact is given by

$$g_O(d)=\frac{(2-\eta)}{2(1-\eta)^3}. \tag{9}$$

At least two alternative expressions for D have been derived in which the Enskog theory is modified to take account of inelasticity [27,28]. However, for the case of reasonably elastic collisions ($\varepsilon=0.91$ in the experiments described here), both modified expressions for D agree with Eq. (8) to within about 10%. We, therefore, use Eq. (8) throughout the analysis described here. Similarly, the convective term in Eq. (6) is given by [25]

$$u(y)=\frac{gd}{16\eta(y)g_O(d)}\left(\frac{E_O(y)}{\pi m}\right)^{-1/2}. \tag{10}$$

IV. SOLUTION OF THE SMOLUCHOWSKI EQUATION

Equation (6) can be solved analytically for the case of an homogeneous bounded system under the influence of gravitational forces [20]. Unfortunately, this solution is unsuitable for a granular bed as the packing fraction and granular temperature profiles result in a complex diffusion profile, the form of which is not known analytically as a function of height. Determination of the packing fraction distributions is reasonably accurate using PEPT [21], but because the temporal resolution of the facility when performing these experiments was only 5 ms, granular temperature profiles could not be determined accurately for anything other than very dilute systems [21]. The approach adopted here has, therefore, to combine an *estimated* $E_O(y)$ with a measured $\eta(y)$, and then to use the resulting diffusion profile $D(y)$ and convection $u(y)$ to solve for $P(y,t)$ by means of a finite-difference scheme. By calculating the time evolution of u at different initial heights, E_O can be modified in an iterative way, as described in Sec. IV E in an attempt to match the experimentally and numerically determined probability densities. This approach serves two purposes: first, it allows the relevance of probabilistic techniques to granular flows to be judged, and second, it provides an estimate of the granular temperature profiles for the experiments assessed.

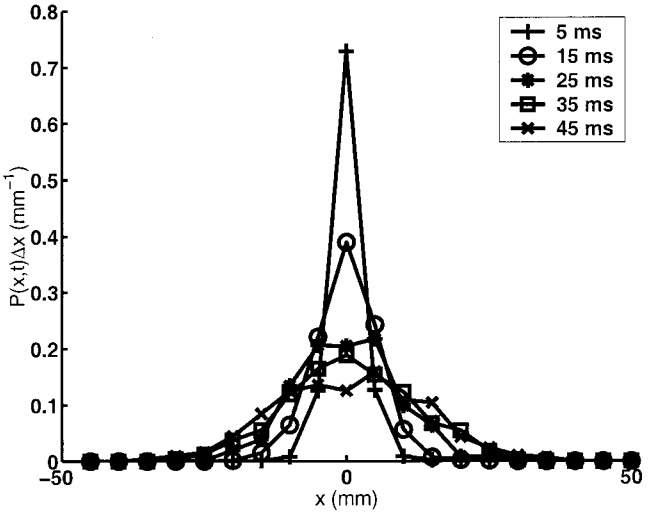

FIG. 2. Displacement probability densities for motion in the x direction, $N=1750$.

Typically the three-dimensional Smoluchowski equation may be solved through separation of the variables. In the x and z directions there are no body forces acting, so Eq. (6) is simplified, with a solution similar to Eq. (3) [see Fig. 2 for examples of experimentally determined curves of $\mathbf{u}(x,t)$]. Therefore it remains to solve the 1D problem for the y direction. The finite-difference formulation, initial and boundary conditions are described analytically in Sec. IV A–IV C. Stability issues are discussed briefly in Sec. IV D, and the iterative procedure used to estimate $E_O(y)$ is presented in Sec. IV E.

A. Finite-difference approximation

A forward time-centered space approach was taken to solve Eq. (6) [29]. A rectangular mesh is generated and the following approximation for nodes away from the boundary regions provides us with a finite-difference approximation for Eq. (6):

$$P(i,j+1)=P(i,j)+\left\{\begin{array}{c}\dfrac{[u(i+1)P(i+1,j)-u(i-1)P(i-1,j)]}{2\Delta y}\\ +\dfrac{D(i+1)[P(i+2,j)-P(i,j)]-D(i-1)[P(i,j)-P(i-2,j)]}{4\Delta y^2}\end{array}\right\}\Delta t+O(\Delta t^2), \quad (11)$$

where i and j are the node positions, and Δy and Δt are the corresponding step sizes between nodes.

B. Initial conditions

At time $t=0$, the probability distribution for a tagged grain is a δ function. Such a function is difficult to represent computationally, so we allow the grains to spread out in a manner given by Eq. (3) until the distribution function is sufficiently wide to be represented accurately on the mesh. This typically corresponds to a time, $t_C \sim 10^{-7}$ s. From this time onward the problem is solved using Eq. (11).

C. Boundary conditions

In a vibrofluidized granular bed the system is bounded by the walls and the base of the experimental cell. Resolving in the y direction means that there is only one boundary to be considered at the base. However, the situation is complicated by the fact that the base actually inputs energy into the system, and is more than just a passive reflecting boundary. A description of such a boundary requires analysis of the velocity as well as the spatial coordinates, leading to a partial differential equation of the form given by the Fokker-Planck equation [20]. This is evidently a considerably more complicated scenario than that described by Eq. (6), and we make the working assumption that the system can be effectively modeled by simply using reflecting boundary conditions, at least for points far from the base.

The fact that grains cannot pass through the base, i.e., $j_Y(y=0)=0$, results in the following Neumann boundary conditions at $y=0$ [23]:

$$D(y)\frac{\partial u(y,t)}{\partial y}+u(y)P(y,t)=0. \quad (12)$$

This boundary condition was approximated by means of an imaginary nodal point positioned one step beyond the base. The no-flux condition can then be expressed in finite-difference form as

$$P(1,j+1)=P(3,j+1)+\frac{2u(2)\Delta y(2,j+1)}{D(2)} \quad (13)$$

where nodal point $i=2$ is the boundary node and $i=1$ is the imaginary point. The five-point finite-difference equation (11) also requires u to be specified at the boundary value $i=2$. This can be expressed using a combination of center and forward-difference approximations, as

$$\begin{aligned}P(2,j+1)=P(2,j)+\frac{\Delta t}{2\Delta y^2}\{&[D(3)-D(2)][P(3,j)\\ &-P(1,j)]+2D(2)[P(3,j)-2P(2,j)+P(1,j)]\\ &+2\Delta y[u(3)P(3,j)-u(2)P(2,j)]\}. \quad (14)\end{aligned}$$

At the top surface no explicit boundary condition is implemented. When the upper limit of the mesh is too small, then significant ''leakage'' out of the system occurs. This leakage was limited by making the height of the mesh at least twice the height of the experimental cell. However, as $y\to\infty$ the packing fraction tends to zero, and hence D diverges, causing the numerical solving routine to break down. This was avoided by forcing the packing fraction to decay asymptotically to a small, but finite value, thus saturating D for $y>0.1$ m.

D. Stability

The analysis of the stability and convergence of the finite-difference approximation to the Smoluchowski equation is a somewhat difficult process due to the varying coefficients, $D(y)$ and $u(y)$. A tentative analysis was carried out for the case of constant D and u. In this case, the approximation is consistent and conditionally stable. We, therefore, make the assumption that there exist combinations of y and t step sizes that mean that Eq. (11) is both convergent and stable. As $P(y,t)$ is a probability density function, then it must satisfy

TABLE I. Granular temperature parameters [see Eq. (16)] for $N=700$, 1750, and 2450.

N	$E_O^{(1)}(10^{-4}\,\mathrm{J})$	$E_O^{(\infty)}(10^{-4}\,\mathrm{J})$	y_E (mm)
700	0.25	0.1	0.1
1750	0.2	0.075	0.1
2450	0.11	0.05	0.1

$$\int_0^\infty P(y,t)dy=1 \tag{15}$$

at each time step. For every solution, this normalization criterion was tested. For each solution that met this criterion, it was assumed that the conditions necessary for both stability and convergence had been met. The results presented in this paper were obtained with step sizes $\Delta y=0.1\,\mathrm{mm}$ and $\Delta t=10^{-9}$ secs.

E. Displacement probability density function

The distributions of D and u in Eq. (6) are unknown and must be estimated. Kinetic theory relations (8) and (10) show that both depend on the packing fraction and granular temperature distributions. The former is measurable experimentally and the latter is estimated in a form of inverse analysis. A simple functional form was chosen based on knowledge of the granular temperature distributions found experimentally and numerically in two dimensions [30,4]. Such studies suggest that the temperature distributions decay in an approximately exponential fashion to some nonzero asymptotic value

$$E_O=(E_O^{(1)}-E_O^{(\infty)})\exp(-y/y_E)+E_O^{(\infty)}, \tag{16}$$

where $E_O^{(\infty)}$ is the asymptotic granular temperature, $E_O^{(1)}$ is the initial value, and y_E specifies the length scale for the decay of E_O. Following the initial estimation of D and $u(y)$ [through Eqs. (8), (10), and (16)] the numerical solution of Eq. (6) is compared with the experimentally determined displacement probability density (DPD). An iterative process is then undertaken in which the three parameters controlling the granular temperature profile are modified until the DPD and the mean squared displacements of the numerical and the experimental DPDs converge. Once this has been achieved, realistic estimates of the $D(y)$ and $u(y)$ distributions can be extracted, and an estimate of the granular temperature profile is known. Table I shows the values of the parameters of Eq. (14) for $N=700$, 1750, and 2450. These parameters show similar trends to those that have been observed in detailed studies of vibrofluidized granular beds [30]. With increasing numbers of grains the initial granular temperature is seen to decrease, complementing the decrease in the value of the asymptotic granular temperature (Fig. 3).

V. RESULTS

A. Displacement probability density function

Due to the variation in the packing fraction between experiments containing different numbers of grains, the time dependent DPDs for each experiment were expected to differ considerably. In a low density granular gas, the grains spread out substantially faster than those in a high density granular gas, due to both the low packing fraction and the relatively high granular temperature [cf. Eq. (8)] [13]. Figures 4(a), 4(b), 5(a), 5(b), 6(a) and 6(b) show the comparison between experimentally determined DPDs and those derived numerically from Eq. (6), for $N=700$, 1750, and 2450, respectively, for two start heights, $y_0=$ (a) 32.5 and (b) 52.5 mm, using the granular temperature profiles defined by the parameters in Table I. The DPDs are compared at intervals of 10 ms–100 ms. The measured packing fraction profiles for these experiments are shown in Fig. 7. These are determined by measuring the residence time distribution of the grain as a function of height, from which the packing fraction profile is calculated using the ergodicity property of the system [21]. At the lower height of 32.5 mm, the correspondence between the numerical and the experimental results is good. For $N=1750$ and 2450 this is true over the whole period of examination [Figs. 4(a) and 5(a)]. For $N=700$, after about 30 ms, the grains begin to collide with the base and discrepancies appear. A no particle flux condition is imposed at the base, but in reality, as well as a no particle flux, there is a net heat flux boundary condition. The Smoluchowski equation is appropriate only for describing the development of the displacement, and any velocity boundary conditions at the base cannot be handled by the equation. Solutions of the problem for $y_0=12.5\,\mathrm{mm}$ were also attempted. The presence of the grains so close to the base meant that discrepancies rapidly became apparent (i.e., at times less than 10 ms) and these results are not shown.

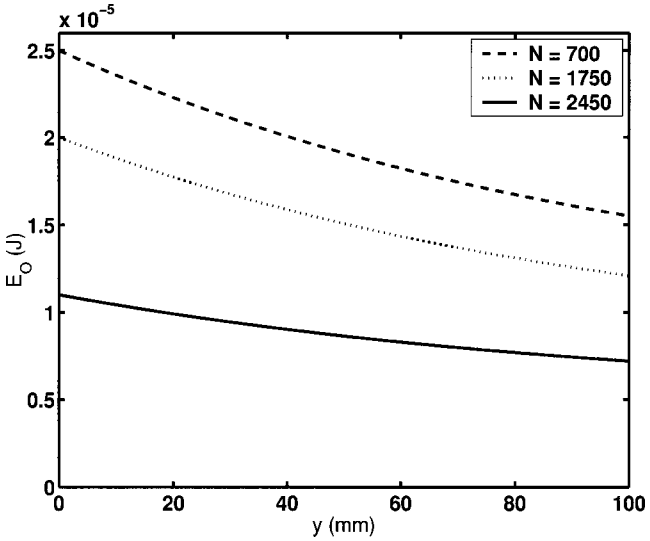

FIG. 3. Granular temperature profiles for $N=700$, 1750, and 2450, determined through successive iteration of the numerical solution of the Smoluchowski equation.

A phenomenological description of an effective terminal velocity $u_{\mathrm{eff}}(y)$ may be estimated through rearrangement of Eq. (5) in the long time limit,

$$u_{\mathrm{eff}}(y)=\frac{-D(y)}{\eta(y)}\frac{d\eta(y)}{dy} \tag{17}$$

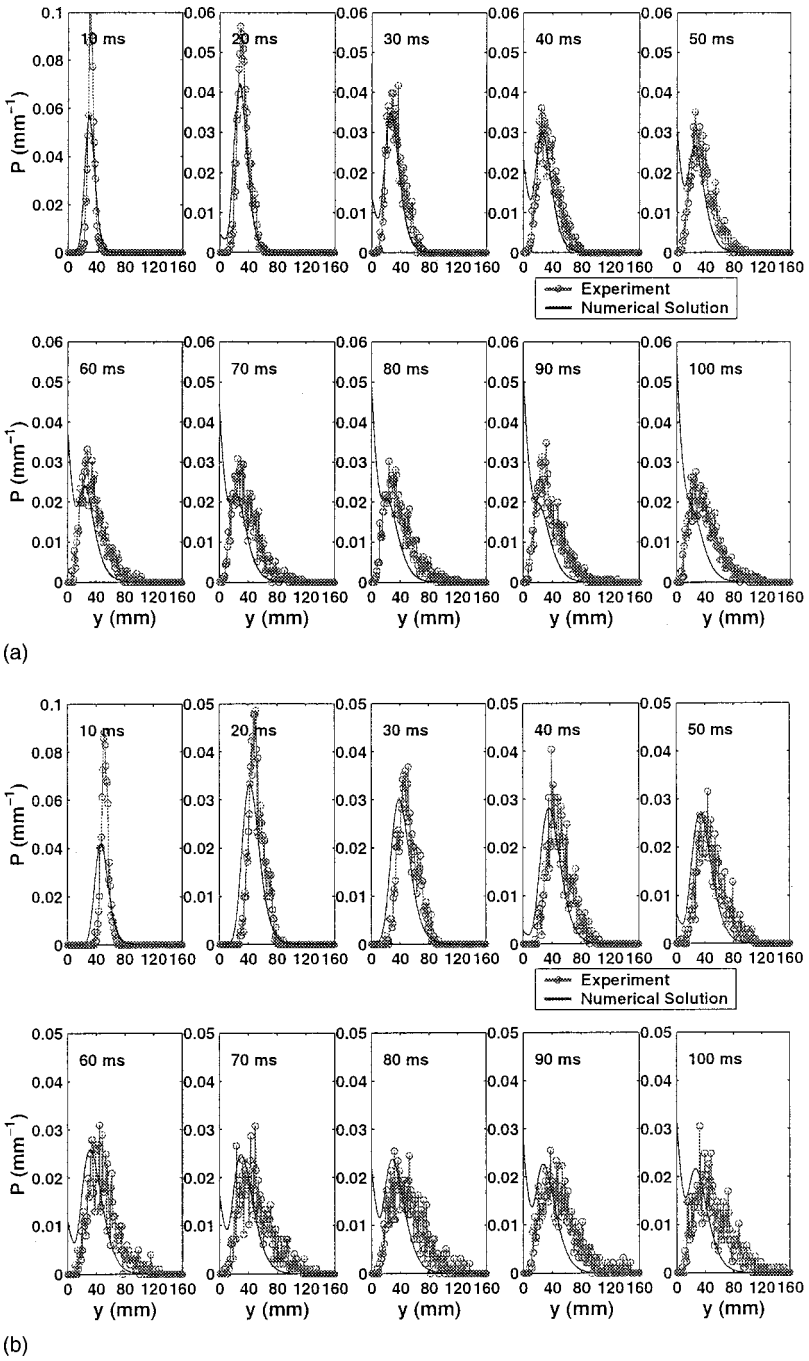

FIG. 4. Comparison of experimentally and numerically determined displacement probability densities. $y_0 =$ (a) 32.5 and (b) 52.5 mm for $N = 700$.

as η is proportional to the steady state displacement probability density, P. In this way, the effect of the vibrating base may be incorporated into $u_{\mathrm{eff}}(y)$ to provide an effective potential in which the grains move. Figure 8 shows a comparison of $u_{\mathrm{eff}}(y)$ [Eq. (17)] and $u(y)$ [Eq. (10)] for $N = 1750$. At locations near to the base, $u_{\mathrm{eff}}(y)$ is negative, i.e., grains are pushed upward away from the lower-most boundary, and consequently we observe low values of packing fraction in this region. At higher altitudes, $u_{\mathrm{eff}}(y)$ converges towards $u(y)$ as the influence of the base is reduced. This implies that, even though the Smoluchowski equation is strictly valid only for the spatial development of grains in a system, it may be extended through the use of an effective potential to describe more complex situations where, for example, energy is being injected into the bed.

Generally, one finds that although the correspondence between theory and experiment is relatively good at long times ($t \sim 100$ ms), it is difficult to replicate the short term behavior. This is most evident at high altitudes, where the packing fraction is low (see Fig. 7) and the mean time between collisions is long. In this state, the system is not diffusive and the motion over long periods is essentially ballistic; the Smoluchowski equation is valid only in a regime where each grain has suffered a large number of collisions, and thus it is to be expected that the model should break down at short times and at large distances from the base.

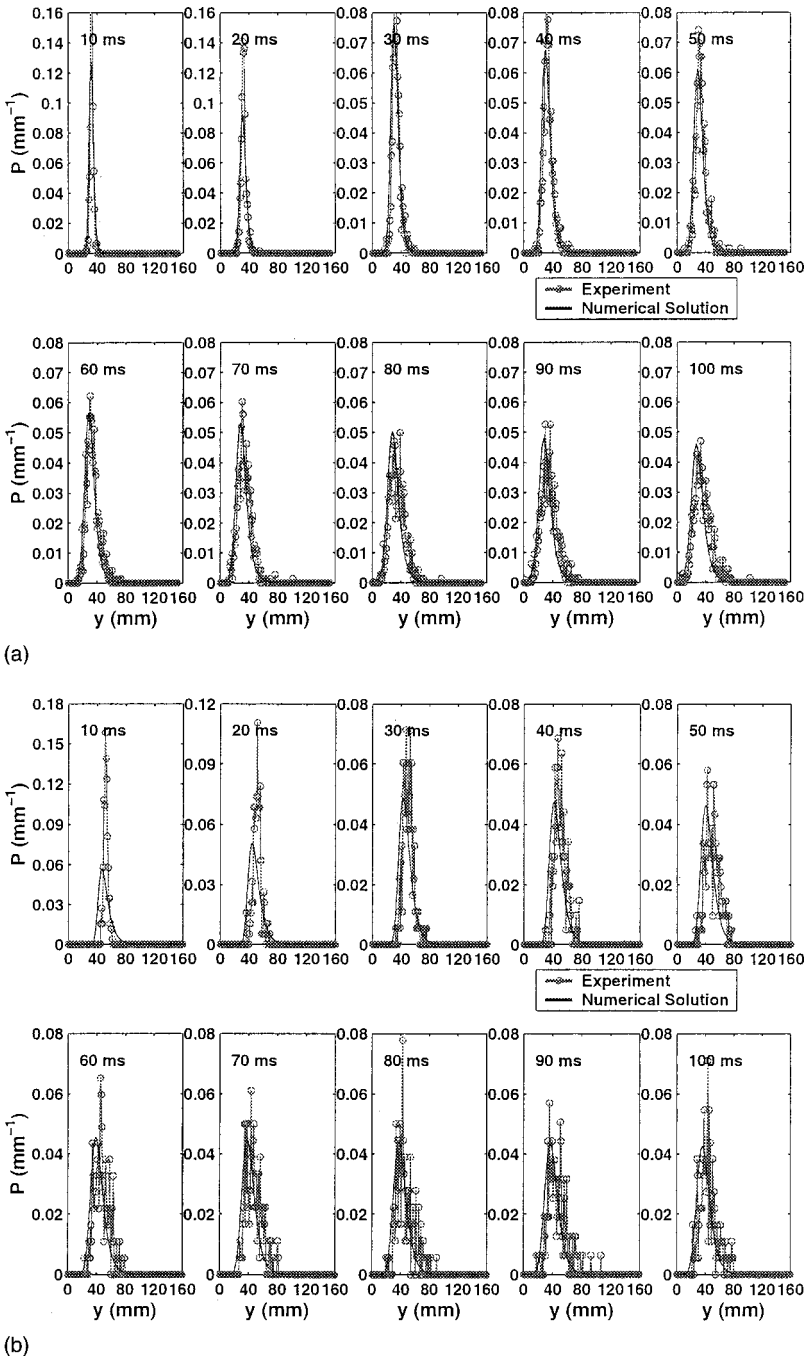

FIG. 5. Comparison of experimentally and numerically determined displacement probability densities. $y_0 =$ (a) 32.5 and (b) 52.5 mm for $N = 1750$.

B. Mean squared displacement

Systems undergoing diffusive behavior are frequently analyzed in terms of the time variation of the mean squared displacement. Departure from the linear behavior predicted by Eq. (4) can provide useful insight into the physical processes occurring on different time scales.

The mean squared displacement can be calculated directly as the second moment of P about the starting point. Figure 9 shows the result for the case $N = 1750$, $y_o =$ (a)32.5 and (b) 52.5 mm. In both plots the agreement between experimental and numerical results is best at long times, with significant discrepancies occurring over the first 20 ms or so, reflecting the results shown in Figs. 5(a) and 5(b). The grains are undergoing ballistic motion over a time scale shorter than the Enskog mean-free time (τ_E), which gives rise to a quadratic toe in the experimental plots [13]. The Smoluchowski equation, on the other hand, models the system as being purely diffusive, even at infinitesimally small times. Poor agreement between experimental and numerical results at times $t < \tau_E$ is therefore not surprising.

Figure 9(a) shows little evidence of the convective behavior incorporated into Eq. (6), i.e., the mean squared displacement reflects solely diffusive motion. For $y_o = 32.5$ mm [Fig. 9(a)] the beads are located close to the maximum packing fraction, and to the mean height of the granular bed. Bulk motion of the grains away from this position will be small;

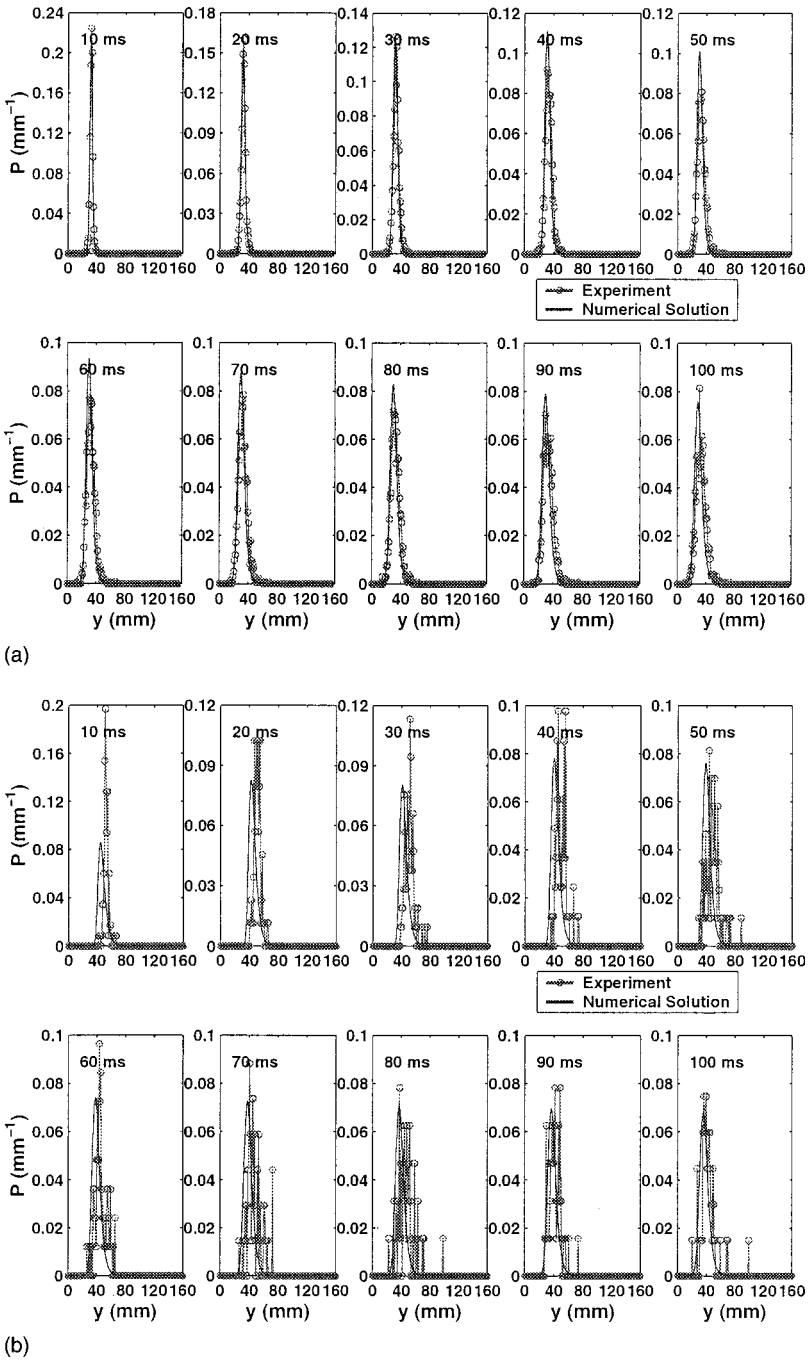

FIG. 6. Comparison of experimentally and numerically determined displacement probability densities. $y_0 =$ (a) 32.5 and (b) 52.5 mm for $N = 2450$.

hence diffusive behavior will dominate the mean squared displacement [Fig. 9(a)]. At high altitudes [Fig. 9(b)] the mean squared displacement shows exaggerated diffusion at short times before crossing over to a value close to that seen experimentally for $t > 20$ ms. This type of behavior is only observed at low η, and is due to nonzero diffusion gradients. Expansion of Eq. (6), considering only the y direction, leads to

$$\frac{\partial P}{\partial t} = \frac{\partial D}{\partial y}\frac{\partial P}{\partial y} + D\frac{\partial^2 P}{\partial y^2} + \frac{\partial (u_Y P)}{\partial y}. \tag{18}$$

The diffusive mode of the granular behavior is controlled by the first two terms on the right hand side of Eq. (18). The first term is only appreciable at short times, when the gradient of P is large. The gradient of the diffusion coefficient, however, is large only at small packing fractions. Therefore, this term is only significant for small η (when $y \geqslant 50$ mm), and at short times, whereupon P becomes skewed [Fig. 5(b)], resulting in an enhanced mean squared displacement of the type seen in Fig. 9(b). This behavior is, of course, not seen experimentally. At short times the grains move ballistically, but at times $t \gg \tau_E$, the gradients of the numerical and experimental mean squared displacements are similar, suggesting the Smoluchowski equation is an effective method of modeling granular flows.

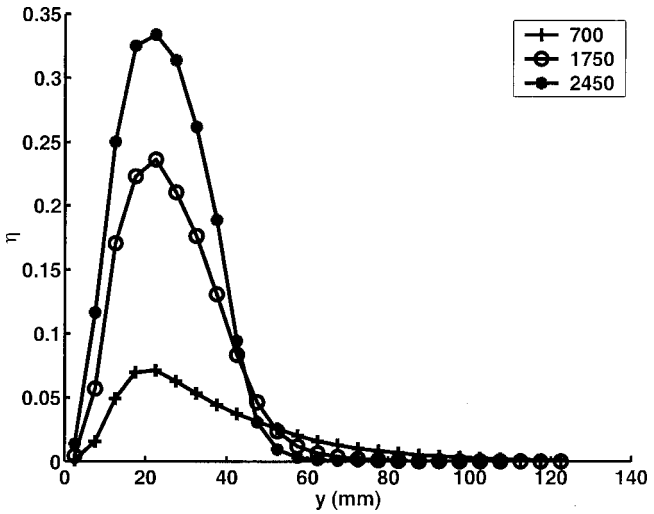

FIG. 7. Packing fraction profiles, $\eta(y)$, for $N=700$, 1750, and 2450; $A_O=1.91$ mm.

VI. CONCLUSIONS

The Smoluchowski equation has been employed to model the spatial development of grains in a vibrofluidized granular bed and has been successfully applied at positions away from the base. The granular temperature profiles resulting from the analysis resemble those seen in two-dimensional studies of vibrated systems, both in overall form and in the order of magnitude of their control parameters. Agreement between the numerical and experimental results was poorest at both very low and very high altitudes. Failure close to the base was attributed to the over-simplistic nature of the boundary conditions employed there. It was found that the behavior of grains close to the base may be described through the use of an effective potential that incorporates the influence of the base as well as gravity on the motion of the grains. Deviations at high altitudes were attributed to increases in mean-free path beyond the dimensions of the cell. Nevertheless, the agreement is perhaps surprisingly good considering the approximations used. It is hoped that the proposed model will provide a useful starting point for the development of both an improved understanding of diffusion within dissipative nonequilibrium systems and of engineering tools to predict mixing behavior in rapid granular flows.

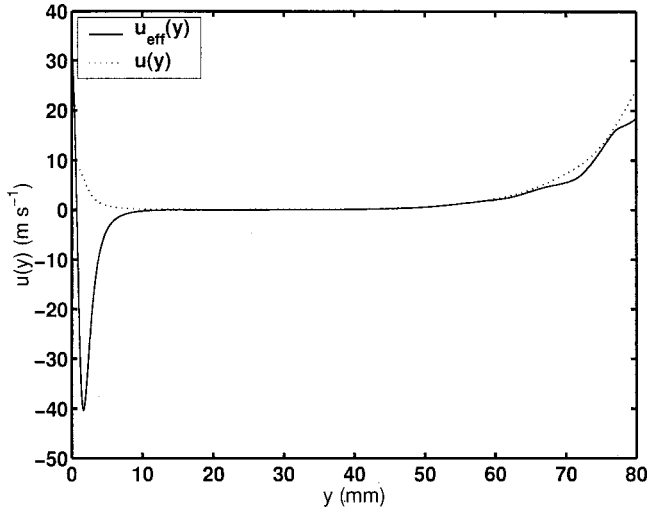

FIG. 8. Comparison of the terminal velocities, $u(y)$ [Eq. (10)] and $u_{\mathrm{eff}}(y)$ [Eq. (17)] for $N=1750$. The effective terminal velocity, $u_{\mathrm{eff}}(y)$, is negative close to the base, implying that grains are pushed away from the vibrating boundary, indicating that grains may be assumed to be moving in an effective potential incorporating both the influence of the vibrating base and of gravity.

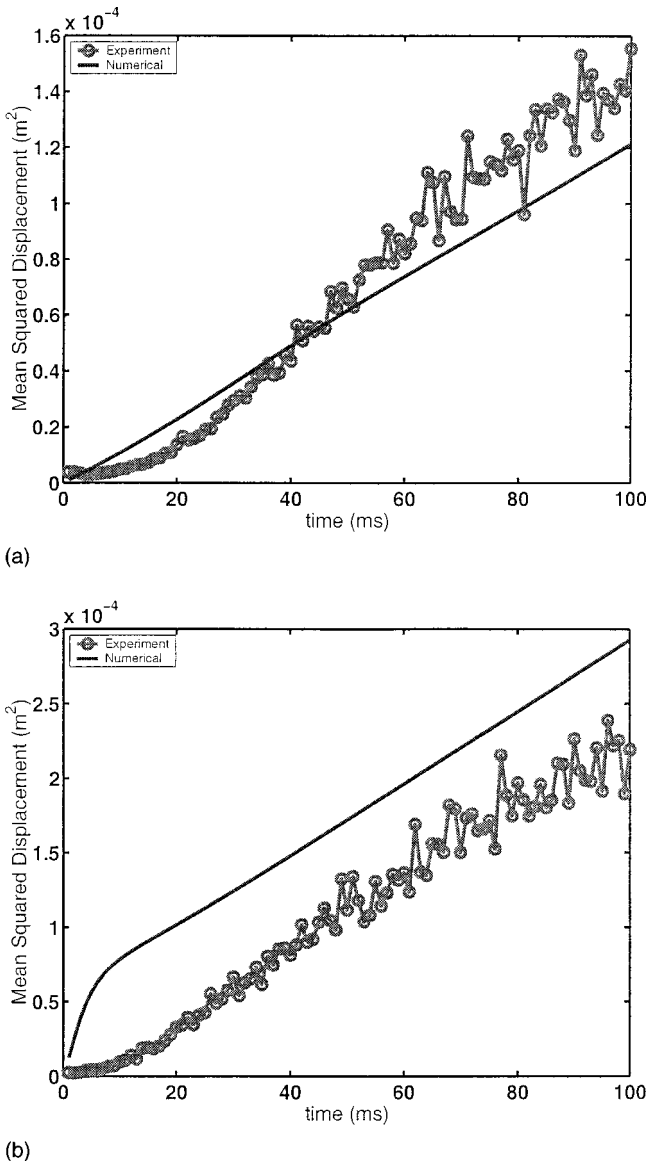

FIG. 9. Mean squared displacement of the grains as a function of time, $y_0=$ (a) 32.5 and (b) 52.5 mm, for $N=1750$.

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